# A short note on the approximability of the Minimum Satisfiability problem

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**Abstract**

*This paper shows that Minimum Satisfiability problem, where it is required to find a truth assignment that satisfies minimum number of clauses, is MAX SNP-hard. Therefore, it is impossible for this problem to have a polynomial-time approximation scheme (PTAS) unless NP=P.*

## 1 Introduction

Satisfiability problem has been studied for quite a long time. It was introduced by Cook and shown to be NP-complete [1]. Therefore, it is believed that there is no efficient algorithm to decide the satisfiability for a given set of clauses. The optimization version of this problem was introduced by Johnson [2], where it is required to find a truth assignment that satisfies the maximum number of clauses. The Maximum Satisfiability (MAX SAT) problem was also proved to be NP-complete. Moreover, it was shown by Papadimitriou to be MAX SNP-complete [3]. Recently, a different optimization version was proposed, where the goal is trying to "minimize" the number of clauses satisfied [4]. In the same paper they proved that Minimum Satisfiability (MIN SAT) problem is NP-complete.

Arora et al. [5] have proved that it's impossible for a MAX SNP-complete problem to have a polynomial-time approximation scheme (PTAS). In this paper, we shall show that MIN SAT is MAN SNP-hard. Hence this tells that it is almost impossible for this problem to have a PTAS.

## 2 Definition

The following definition is given in [3].

**Definition 1** *Let $A$ and $B$ be two optimization (maximization or minimization) problems. We say that $A$* L-reduces *to $B$ if there are two polynomial-time transformation $f$, $g$, and constants $\alpha, \beta > 0$, such that for each instance $x$ of $A$, the following properties hold:*

1. *$f(x)$ is an instance of $B$ such that the optima of $x$ and $f(x)$, denoted as $opt_A(x)$ and $opt_B(f(x))$, respectively, satisfy*

$$opt_B(f(x)) \leq \alpha opt_A(x)$$

2. *Given any solution $y$ of $f(x)$ in $B$ with cost $c_B(f(x), y)$, the transformation $g$ produces a solution of $x$ with cost $c_A(x, g(x, y))$ such that $|c_A(x, g(x, y)) - opt_A(x)| \leq \beta |c_B(f(x), y) - opt_B(f(x))|$*

**Definition 2** *Let $A$ be an optimization problem. $\forall x \in A$, let $c$ and $opt_A(x)$ denote the cost of the approximation solution and the optimum solution, respectively. The* error-ratio *of this approximation solution is defined to be*

$$\frac{|c - opt_A(x)|}{opt_A(x)}$$

**Proposition 1** *If $A$ L-reduces to $B$, and there is a polynomial-time approximation algorithm for $B$ with worst-case error-ratio $\epsilon$, then there is a polynomial-time approximation algorithm for $A$ with worst-case error-ratio $\alpha\beta\epsilon$.*

**Proof.** $A$ L-reduces to $B$, so by the definition we have

1. $opt_B(f(x)) \leq \alpha opt_A(x)$, and

2. $|c_A(x, g(x, y)) - opt_A(x)| \leq$

   $\beta |c_B(f(x), y) - opt_B(f(x))|$

   Therefore, the error-ratio for $g(x, y)$ is

$$\frac{|c_A(x,g(x,y)) - opt_A(x)|}{opt_A(x)} \leq \frac{\beta |c_B(f(x),y) - opt_B(f(x))|}{\frac{opt_B(f(x))}{\alpha}}$$

$$= \alpha\beta \frac{|c_B(f(x), y) - opt_B(f(x))|}{opt_B(f(x))}$$

$$= \alpha\beta\epsilon$$

## 3 Problem Definition

**Definition 3 (2-MINSAT)** *Given*

1. *A set $U$ of $k$ variables,*

2. *A collection $C$ of $n$ clauses over $U$ such that each $c \in C$ has $|c| = 2$ literals.*

Goal: *Find a truth assignment for $U$ that satisfies minimum number of clauses in $C$?*

**Definition 4 (2-MAXSAT)** *Given*

1. *A set $V$ of $h$ variables,*

2. *A collection $D$ of $l$ clauses over $V$ such that each $d \in D$ has $|d| = 2$ literals.*

Goal: *Find a truth assignment for $V$ that satisfies maximum number of clauses in $D$?*

## 4 The L-reduction

We now define the 4-tuple $(f, g, \alpha, \beta)$ which constitutes the L-reduction from 2-MAXSAT (which we shall call problem A in brief) to 2-MINSAT (which we call problem B). This L-reduction is similar to the one which was adopted in [4] to show the NP-completeness of MINSAT.

The transformation $f$ is defined as follows: For each clause

$$d = q_a \vee q_b$$

where $q_a$ and $q_b$ are literals, transform it to become two clauses

$$\bar{q_a} \vee w_d, \quad \bar{q_b} \vee \bar{w_d}$$

Thus, we transform a 2-MAXSAT problem with $h$ variables, $l$ clauses to a 2-MINSAT problem with $h + l$ variables and $2l$ clauses.

The transformation $g$ from a truth assignment $(x_1, x_2, \ldots, x_h, w_1, w_2, \ldots, w_l)$ in $B$ back to $A$ is very simple – just dropping those auxiliary variables $w_i$ to become $(x_1, x_2, \ldots, x_h)$.

**Proposition 2** *In [4], it is shown that such a transformation guarantees that* $m$ clauses are satisfied in 2-MAXSAT if and only if $2l - m$ clauses clauses are satisfied in 2-MINSAT.

In the following, we shall determine the two constants $\alpha$ and $\beta$ to construct the L-reduction.

**Lemma 1** *Let $x$ be an instance of 2-MAXSAT, and $m^* = opt_A(x)$. Then $m^* \geq \frac{l}{2}$, where $l$ is the number of clauses.*

**Proof.** There is a simple heuristic algorithm that can satisfy at least half of the clauses. Let $C$ be initialized to be a set containing all clauses,

```
while C in not empty do
    begin
    Pick a variable x_i whose truth assignment
        hasn't been determined
    Count the occurrence of x_i and x̄_i
    if x_i appears in more clauses than x̄_i does
        then set x_i to be TRUE
        else set x_i to be FALSE
    Remove the clauses that contains
        literals x_i or x̄_i
    end.
```

It can be easily seen that at each removal, at least half of the removed clauses are satisfied. Therefore at the termination of this algorithm, totally at least $\frac{l}{2}$ clauses will be satisfied. Hence it follows that the optimum $m^* \geq \frac{l}{2}$. $\square$

**Lemma 2** *There exists $\alpha > 0$ such that $\forall x \in A, opt_B(f(x)) \leq \alpha opt_A(x)$.*

**Proof.** Let $opt_A(x) = m^*$. By Proposition 1, we have $opt_B(f(x)) = 2l - m^*$. From Lemma 1, $m^* \geq \frac{l}{2}$. Therefore, if we set $\alpha = 3$, then $opt_B(f(x)) - \alpha opt_A(x) = 2l - m^* - 3m^* = 2(l - 2m^*) \leq 0$ $\Rightarrow opt_B(f(x)) \leq \alpha opt_A(x)$. $\square$

**Lemma 3** *There exists $\beta > 0$ such that for each truth assignment $y \in B$, $|opt_A(x) - c_A(x, g(x,y))| \leq \beta |c_B(f(x), y) - opt_B(f(x))|$.*

**Proof.** Let $2l-m$ be the number of clauses satisfied by the truth assignment $y$, i.e. $c_B(f(x), y) = 2l - m$, and $opt_B(f(x)) = 2l - m^*$. From the construction of our reduction, it follows that $opt_A(x) = m^*$ and $c_A(x, g(x, y)) = m$. Hence, $|opt_A(x) - c_A(x, g(x, y))| = |c_B(f(x), y) - opt_B(f(x))| = |m^* - m|$. Setting $\beta = 1$ will make the desired inequality hold. $\square$

In [3] it was proved that 2-MAXSAT is MAX SNP-complete. Therefore, From the above lemmas and the definition of MAX SNP-hardness, we can obtains the following theorem:

**Theorem 1**

1. 2-MINSAT is MAX SNP-hard, since 2-MAXSAT L-reduces to 2-MINSAT.

2. MINSAT is also MAX SNP-hard, for 2-MINSAT is only a special case of it.

## 5 Conclusion and Further Study

By showing that a MAX SNP-complete problem (Maximum Satisfiability problem) L-reduces to Minimum Satisfiability problem, we've proved that the problem MIN SAT is MAX SNP-hard. That is, it is believed that MIN SAT cannot have a polynomial-time approximation scheme (PTAS). However, we haven't determined whether MIN SAT belongs to the class MAX SNP or not. If so, we can even claim that this problem is MAX SNP-*complete*. This deserves further study.

## References

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